

# THE ARCLENGTH OF A RANDOM LEMNISCATE IN THE PLANE

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## 1. INTRODUCTION

A (polynomial) lemniscate is a curve defined in the complex plane by the equation  $|p(z)| = t$ , where  $p$  is a polynomial of degree  $n$ . From the conjugation-invariant equation  $p(z)\overline{p(z)} = t^2$ , it is apparent that the lemniscate is a real algebraic curve of degree  $2n$ . Calculating the length of a lemniscate is a problem of classical Mathematics that played a role in the development of elliptic integrals. Namely, the length of Bernoulli's lemniscate  $|z^2 - 1| = 1$  is an elliptic integral of the second kind (the same integral appears in classical mechanics, as the period of a pendulum, and in classical statics, as the length of an elastica).

**1.1. The Erdős lemniscate problem.** Erdős, Herzog, and Piranian [3] posed the extremal problem of determining the maximum length of a lemniscate

$$\Lambda = \{z \in \mathbb{C} : |p(z)| = 1\}$$

when  $p$  is a monic polynomial of degree  $n$ . The problem was restated by Erdős several times (e.g., see [4]) and is often referred to as the *Erdős lemniscate problem*. Taking  $p$  monic guarantees that the length of the lemniscate is bounded, for instance by  $2\pi n$  [1]. The maximum was conjectured [3] to occur for the so-called *Erdős lemniscate*,  $\{z \in \mathbb{C} : p(z) = z^n - 1\}$ ; Fryntov and Nazarov [6] have proved that this curve is indeed a *local* maximum and that as  $n \rightarrow \infty$  the maximum length is  $2n + o(n)$  which is asymptotic to the conjectured extremal.

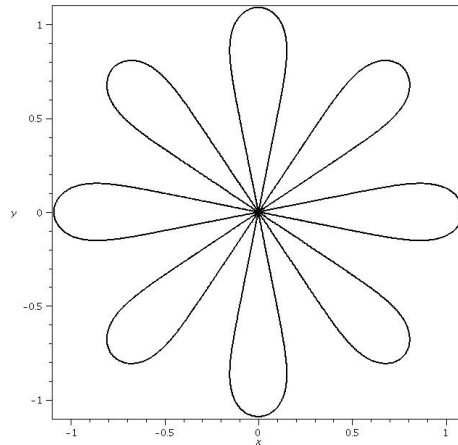


FIGURE 1. The Erdős lemniscate for  $n = 8$ .

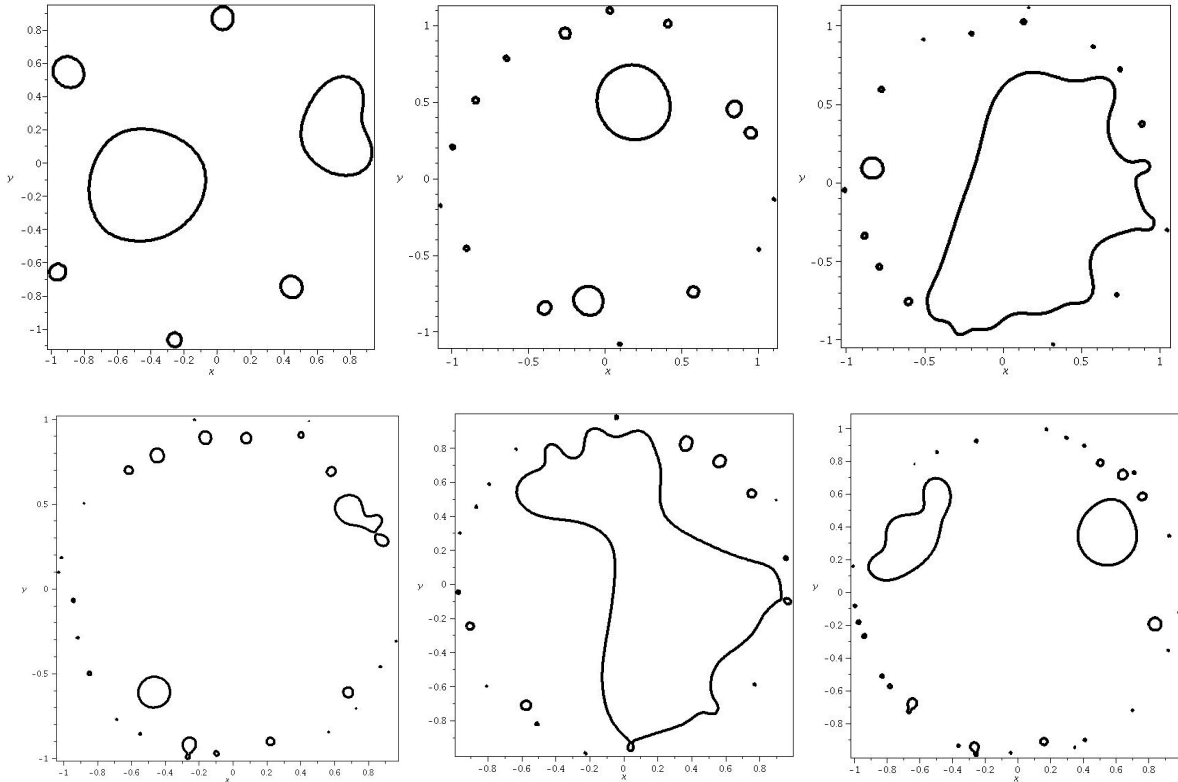


FIGURE 2. Random lemniscates using Kac polynomials of degree  $n = 10, 20, 30, 40, 50, 60$  (from left to right).

**1.2. The arclength of a random lemniscate.** Motivated by seeking a broad point of view on the Erdős lemniscate problem, we give a probabilistic treatment of the length, by studying the *average* arclength of a random polynomial lemniscate.

We select  $\Lambda = \Lambda_n$  randomly by taking  $p_n(z)$  to be a random polynomial from the Kac ensemble, that is,

$$(1) \quad p_n(z) = \sum_{k=0}^n a_k z^k,$$

where  $a_k \sim N_{\mathbb{C}}(0, 1)$  are independent, identically distributed complex Gaussians. The resulting distribution for the random curve  $\Lambda$  is invariant under rotation of the angular coordinate. Indeed, we have:

$$|p_n(e^{i\theta} z)| = \left| \sum_{k=0}^n a_k e^{ik\theta} z^k \right|,$$

and invariance follows from the observation that  $b_k = a_k e^{ik\theta}$  are i.i.d and distributed as  $N_{\mathbb{C}}(0, 1)$ .

The Kac ensemble is one of the most well-studied random polynomials, and it seems especially appropriate in the context of the Erdős lemniscate problem, since the zeros of  $p_n$  resemble those of the defining polynomial of the Erdős lemniscate in that they are approximately equidistributed on the unit circle [11, 12].

We now state our main result.

**Theorem 1.** *Consider a sequence of random polynomials  $p_n(z) = \sum_{k=0}^n a_k z^k$ , where the  $a_k$  are i.i.d  $N_{\mathbb{C}}(0, 1)$ . Let  $\Lambda_n = \{z \in \mathbb{C} : |p_n(z)| = 1\}$ . Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}|\Lambda_n| = C,$$

where the constant  $C \approx 8.3882$  is given by the integral (11) below.

**1.3. The Erdős lemniscate is an outlier.** The following Corollary of Theorem 1 provides weak concentration of measure around lemniscates having length of constant order.

**Corollary 2.** *Let  $L_n$  be any sequence with  $L_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The probability that  $|\Lambda_n| \geq L_n$  converges to zero.*

*Proof.* Since the length  $|\Lambda_n|$  is a positive random variable, we can apply Markov's inequality:

$$\mathbb{P}\{|\Lambda_n| \geq L_n\} \leq \frac{\mathbb{E}|\Lambda_n|}{L_n} = O(L_n^{-1}), \quad \text{as } n \rightarrow \infty,$$

by Theorem 1. □

In particular, the probability that the length has the same order as the extremal case (i.e., exceeding some fixed portion of  $n$ ) converges to zero. The authors expect that this rate of decay can be improved, and it would be interesting to investigate this topic from the point of view of large deviations.

**1.4. Remarks.** The Erdős lemniscate is extremely singular and symmetric (see Figure 1), and its length appears to diminish rapidly under perturbations. Naively, this suggests that it occupies a rather far corner of the parameter space. The probabilistic approach taken here provides a framework for articulating a precise formulation of this intuitive notion as we have done in Section 1.3.

The outcome for the average length of a random lemniscate depends on the definition of “random”. The Kac model seems most appropriate in the current context. We consider several models in the sections below, including the case that the variances have binomial coefficient weights and also the case in which they have reciprocal binomial coefficient weights. In each of these cases, the expected length has order  $O(n^{-1/2})$ .

Another extremal problem, to find the maximal spherical length of a rational lemniscate, was posed and solved by Eremenko and Hayman [5]. Lerario and the first author considered random rational lemniscates on the Riemann sphere [10], and studied such statistics as spherical length, number of components, and arrangement (nesting) of components.

While nesting of components is not possible for polynomial lemniscates (which follows from the maximum principle), it would still be interesting to study the connected components of  $\Lambda$ . In particular, the samples plotted in Figure 2 suggest that there is a positive probability (independent of  $n$ ) of having a “giant component” that accounts for at least some fixed portion of the total length of  $\Lambda$ .

**1.5. Outline of the paper.** Theorem 1 will follow from a more general result proved in Section 2, namely, Theorem 4 provides the expected length while allowing the coefficients appearing in (1) to be independent centered Gaussians with different variances. The methods in proving Theorem 4 are based on planar integral geometry combined with the Kac-Rice formula. In Section 3, we then derive Theorem 1 as a consequence of Theorem 4. We also apply Theorem 4 to three other models: lemniscates generated by Kostlan polynomials are treated in Section 4, Weyl polynomials in Section 5, and a model that we call the “reciprocal binomial” model is considered in Section 6.

## 2. A MORE GENERAL LENGTH FORMULA FOR GAUSSIAN POLYNOMIALS

In this section we merely assume that the coefficients appearing in  $p(z)$  are centered, independent, but not necessarily identically distributed complex Gaussians.

**2.1. Length and integral geometry.** Applying the integral geometry formula as in [5], we have:

$$|\Lambda_n| = \frac{1}{2} \int_0^\pi \int_{-\infty}^\infty N_n(\theta, y) d\theta dy,$$

where  $N_n(\theta, y)$  is the number of intersections of  $\Lambda_n$  with the line  $L(\theta, y) := \{z \in \mathbb{C} : \Im(e^{-i\theta} z) = y\}$ . Taking the expectation of both sides and using the rotational invariance of  $\Lambda_n$ , we have:

$$(2) \quad \mathbb{E}|\Lambda_n| = \frac{1}{2} \int_0^\pi \int_{-\infty}^\infty \mathbb{E}N_n(\theta, y) d\theta dy = \frac{\pi}{2} \int_{-\infty}^\infty \mathbb{E}N_n(0, y) dy.$$

**2.2. The Kac-Rice formula.** We use the Kac-Rice formula to compute  $\mathbb{E}N_n(0, y)$  which equals the average number of real zeros of the function

$$p_n(z) \overline{p_n(z)} - 1,$$

restricted to the line  $L(0, y)$ . We have:

$$\frac{\partial}{\partial x} (p_n(z) \overline{p_n(z)} - 1) = p'_n(z) \overline{p_n(z)} + p_n(z) \overline{p'_n(z)}.$$

Applying the Kac-Rice formula, we have:

$$(3) \quad \mathbb{E}N_n(0, y) = \int_{-\infty}^\infty \mathbb{E} \delta(|p_n(z)|^2 - 1) |p'_n(z) \overline{p_n(z)} + p_n(z) \overline{p'_n(z)}| dx.$$

For the sake of notational clarity we will henceforth suppress the dependence on  $n$ . So for instance  $\Lambda_n$  will be denoted by  $\Lambda$ ,  $p_n$  by  $p$  etc. We can rewrite (3) in terms of the Gaussian random complex vector  $(U, V) = (p(z), p'(z))$  whose joint probability density function is:

$$\rho(u, v; x + iy) = \frac{1}{\pi^2 |\Sigma|} \exp\{-(u, v)^* \Sigma^{-1} (u, v)\},$$

where  $\Sigma$  is the covariance matrix of  $(U, V) = (p(z), p'(z))$ , which can be computed explicitly using the covariance kernel  $K(z, w)$ :

$$K(z, w) = \mathbb{E} p(z) \overline{p(w)}.$$

Namely, we have:

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

where

$$(4) \quad a = K(z, z), b = \partial_z K(z, z) \text{ and } c = \partial_z \partial_{\bar{z}} K(z, z).$$

In terms of this joint density, the expectation inside (3) can be expressed as:

$$\begin{aligned} \mathbb{E} \delta(|p(z)|^2 - 1) |p'(z) \overline{p(z)} + p(z) \overline{p'(z)}| &= \int_{\mathbb{C}} \int_{\mathbb{C}} \delta(|u|^2 - 1) |v \bar{u} + u \bar{v}| \rho(u, v; x + iy) dA(v) dA(u) \\ &= \int_{|u|=1} \int_{\mathbb{C}} \frac{1}{2|u|} |v \bar{u} + u \bar{v}| \rho(u, v; x + iy) dA(v) dA(u) \\ &= \frac{1}{2} \int_{|u|=1} \int_{\mathbb{C}} |v \bar{u} + u \bar{v}| \rho(u, v; x + iy) dA(v) dA(u), \end{aligned}$$

where we have used the composition property of the  $\delta$ -function ([7], Chapter 6) allowing integration against  $\delta(|u|^2 - 1)$  to be replaced by an integration along the set  $|u|^2 = 1$ .

For  $|u| = 1$ , we notice that

$$\begin{aligned}\rho(u, v; z) &= \frac{1}{\pi^2 |\Sigma|} \exp\{-u\bar{u}(1, \bar{u}v)^* \Sigma^{-1}(1, \bar{u}v)\} \\ &= \frac{1}{\pi^2 |\Sigma|} \exp\{-(1, \bar{u}v)^* \Sigma^{-1}(1, \bar{u}v)\} \\ &= \rho(1, \bar{u}v; z).\end{aligned}$$

Making the change of variables  $t = \bar{u}v$ ,  $dA(t) = dA(v)$ , the integral above becomes

$$\frac{1}{2} \int_{|u|=1} \int_{\mathbb{C}} |t + \bar{t}| \rho(1, t; x + iy) dA(t) du = \pi \int_{\mathbb{C}} |t + \bar{t}| \rho(1, t; x + iy) dA(t).$$

Thus, we have:

$$\mathbb{E}N(0, y) = 2\pi \int_{-\infty}^{\infty} \int_{\mathbb{C}} |\Re\{t\}| \rho(1, t; x + iy) dA(t) dx.$$

Inserting this into the integral geometry formula (2) gives:

$$(5) \quad \mathbb{E}|\Lambda| = \pi^2 \int_{\mathbb{C}} \int_{\mathbb{C}} |\Re\{t\}| \rho(1, t; z) dA(t) dA(z).$$

Observe that the density  $\rho$  can be factored:

$$\begin{aligned}\rho(1, t; z) &= \frac{\exp\{-\frac{1}{a}\}}{\pi a} \frac{a}{\pi |\Sigma|} \exp\left\{-\frac{a}{|\Sigma|} \left|t - \frac{b}{a}\right|^2\right\} \\ &= \frac{\exp\{-\frac{1}{a}\}}{\pi a} \hat{\rho}(t),\end{aligned}$$

where  $\hat{\rho}$  is the probability density function for a complex Gaussian  $N_{\mathbb{C}}(\mu, \sigma^2)$  with mean  $\mu = b/a$  and variance  $\sigma^2 = \frac{|\Sigma|}{a}$ . Thus, the following lemma applies.

**Lemma 3.** *Let  $\zeta \sim N_{\mathbb{C}}(\mu, \sigma^2)$  be a complex Gaussian with mean  $\mu = \mu_1 + i\mu_2$ . Then the absolute moment  $\mathbb{E}|\zeta_1|$  of the real part of  $\zeta = \zeta_1 + i\zeta_2$  is given by*

$$\mathbb{E}|\zeta_1| = \frac{\sigma}{\sqrt{\pi}} \exp\{-\mu_1^2/\sigma^2\} + |\mu_1| \operatorname{erf}(|\mu_1|/\sigma).$$

*Proof of Lemma 3.* We have

$$\begin{aligned}\mathbb{E}|\zeta_1| &= \frac{1}{\pi \sigma^2} \int_{\mathbb{C}} |\zeta_1| \exp\left\{-\frac{|\zeta - \mu|^2}{\sigma^2}\right\} dA(\zeta) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} |\sigma w_1 + \mu_1| \exp\{-|w|^2\} dA(w),\end{aligned}$$

where we have made the change of variables  $w = \frac{\zeta - \mu}{\sigma}$ ,  $dA(w) = \frac{1}{\sigma^2} dA(\zeta)$ .

Letting  $H := \{w \in \mathbb{C} : \sigma w_1 + \mu_1 > 0\}$ , we can rewrite the above integral as:

$$\frac{1}{\pi} \left( \int_H (\sigma w_1 + \mu_1) \exp\{-|w|^2\} dw_1 dw_2 - \int_{\mathbb{C} \setminus H} (\sigma w_1 + \mu_1) \exp\{-|w|^2\} dw_1 dw_2 \right).$$

Since  $\sigma w_1$  is odd and  $\mu_1$  is even (with respect to  $w_1$ ) this can be rewritten as:

$$(6) \quad \frac{1}{\pi} \left( \int_R |\mu_1| \exp\{-|w|^2\} dw_1 dw_2 + \sigma \int_{\mathbb{C} \setminus R} |w_1| \exp\{-|w|^2\} dw_1 dw_2 \right),$$

where  $R := \left\{ w \in \mathbb{C} : |w_1| < \frac{|\mu_1|}{\sigma} \right\}$ . The first integral can be computed in terms of the error function, erf:

$$(7) \quad \int_R |\mu_1| \exp\{-|w|^2\} dw_1 dw_2 = \pi |\mu_1| \operatorname{erf}(|\mu_1|/\sigma),$$

and the second integral is elementary:

$$(8) \quad \int_{\mathbb{C} \setminus R} |w_1| \exp\{-|w|^2\} dw_1 dw_2 = \sqrt{\pi} \exp\{-\mu_1^2/\sigma^2\}.$$

Collecting (6), (7), and (8), we arrive at the formula stated in the lemma.  $\square$

Applying Lemma 3 to (5), we obtain the following main result of this section:

**Theorem 4.** *Let  $p(z)$  be a random polynomial whose coefficients are independent centered Complex Gaussians. Then the expected length of its lemniscate  $\Lambda := \{z \in \mathbb{C} : |p(z)| = 1\}$  is given by*

$$(9) \quad \mathbb{E}|\Lambda| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[ \sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

where as above  $|\Sigma|$  denotes the determinant of the covariance matrix  $\Sigma$  and, the terms  $a, b, c$  are the entries of  $\Sigma$  given by (4).

### 3. KAC POLYNOMIALS: PROOF OF THEOREM 1

In the case  $p(z)$  is a random Kac polynomial, for the entries in the covariance matrix,

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

we have

$$\begin{aligned} a &= K(z, z) = \sum_{k=0}^n |z|^{2k}, \\ b &= \partial_z K(z, z) = \bar{z} \sum_{k=1}^n k |z|^{2k-2}, \\ c &= \partial_z \partial_{\bar{z}} K(z, z) = \sum_{k=1}^n k^2 |z|^{2k-2}. \end{aligned}$$

We will show that the pointwise limit of the integrand appearing in (9) as  $n \rightarrow \infty$  is:

$$(10) \quad \begin{cases} \exp\{-(1-|z|^2)\} \left[ \frac{\exp\{-x^2(1-|z|^2)\}}{(1-|z|^2)^{1/2}} + \sqrt{\pi} x \operatorname{erf}\left\{x\sqrt{1-|z|^2}\right\} \right], & |z| < 1, \\ 0, & |z| \geq 1. \end{cases}$$

We will also show that the dominated convergence theorem applies, so that the integral in Theorem 4 has a limit as  $n \rightarrow \infty$  given by the integral of (10). After changing to polar coordinates, this becomes:

$$\begin{aligned}
 (11) \quad C &:= \lim_{n \rightarrow \infty} \mathbb{E}|\Lambda| \\
 &= \sqrt{\pi} \int_{|z| < 1} \exp\{-(1 - |z|^2)\} \left[ \frac{\exp\{-x^2(1 - |z|^2)\}}{(1 - |z|^2)^{1/2}} + \sqrt{\pi}x \operatorname{erf}\{x\sqrt{1 - |z|^2}\} \right] dA(z) \\
 &\approx 8.3882,
 \end{aligned}$$

which proves Theorem 1. It remains to compute the pointwise limit and to show dominated convergence.

First, we derive certain formulas from the covariance kernel  $K(z, w)$  of the Kac polynomial,

$$K(z, w) = \mathbb{E}p(z)\overline{p(w)} = \sum_{k=0}^n (z\bar{w})^k = \frac{1 - (z\bar{w})^{n+1}}{1 - z\bar{w}}.$$

Notice that

$$a = K(z, z) = \sum_{k=0}^n |z|^{2k} = \frac{1}{1 - |z|^2} - \frac{|z|^{2n+2}}{1 - |z|^2}.$$

We have

$$\frac{\Re\{b\}}{a} = \Re\{\partial_z \log K(z, z)\} = \frac{x}{(1 - |z|^2)} - \frac{(n+1)x|z|^{2n}}{1 - |z|^{2n+2}},$$

and from this we observe that for  $|z| < 1$ ,

$$\begin{aligned}
 \frac{\Re\{b\}}{a^2} &= x \frac{\frac{1}{1 - |z|^2} - \frac{(n+1)|z|^{2n}}{1 - |z|^{2n+2}}}{\frac{1}{1 - |z|^2} - \frac{|z|^{2n+2}}{1 - |z|^2}} \\
 &= x \frac{1 - (n+1)\frac{|z|^{2n}}{\sum_{k=0}^n |z|^{2k}}}{1 - |z|^{2n+2}} \\
 &< x,
 \end{aligned}$$

and as  $n \rightarrow \infty$ ,  $\frac{\Re\{b\}}{a^2} \rightarrow x$ .

On the other hand for  $|z| > 1$ , we note that

$$\begin{aligned}
 \frac{\Re\{b\}}{a^2} &= x \frac{\frac{(n+1)|z|^{2n}}{|z|^{2n+2}-1} - \frac{1}{|z|^2-1}}{\frac{|z|^{2n+2}}{|z|^2-1} - \frac{1}{|z|^2-1}} \\
 &= x \frac{\frac{(n+1)|z|^{2n}}{\sum_{k=0}^n |z|^{2k}} - 1}{|z|^{2n+2} - 1} \\
 &< x,
 \end{aligned}$$

and as  $n \rightarrow \infty$ ,  $\frac{\Re\{b\}}{a^2} \rightarrow 0$ . Keeping in mind to apply the dominated convergence theorem for  $|z| > 1$ , we estimate as follows.

$$\left| \frac{\Re\{b\}}{a^2} \right| \leq |x|, \quad 1 < |z| < 2$$

$$\left| \frac{\Re\{b\}}{a^2} \right| \leq |x| \frac{2n}{|z|^{2n+2}} \leq \frac{2}{|z|^3}, \quad |z| > 2$$

From

$$\frac{|\Sigma|}{a^2} = \partial_{\bar{z}} \partial_z \log K(z, z) = \frac{1}{(1 - |z|^2)^2} - \frac{(n+1)^2 |z|^{2n}}{(1 - |z|^{2n+2})^2},$$

we notice that for  $|z| < 1$ ,

$$\begin{aligned} \frac{|\Sigma|}{a^3} &= \frac{\partial_{\bar{z}} \partial_z \log K(z, z)}{K(z, z)} \\ &= \frac{1}{1 - |z|^2} \left( \frac{1 - \frac{(n+1)^2 |z|^{2n}}{(\sum_{k=0}^n |z|^{2k})^2}}{1 - |z|^{2n+2}} \right) \\ &< \frac{1}{1 - |z|^2}, \end{aligned}$$

and  $\frac{|\Sigma|}{a^3} \rightarrow \frac{1}{1 - |z|^2}$  as  $n \rightarrow \infty$ .

A similar computation for  $|z| > 1$ , yields

$$\frac{|\Sigma|}{a^3} = \frac{1}{|z|^2 - 1} \left( \frac{1 - \frac{(n+1)^2 |z|^{2n}}{(\sum_{k=0}^n |z|^{2k})^2}}{|z|^{2n+2} - 1} \right) < \frac{1}{|z|^2 - 1},$$

and as  $n \rightarrow \infty$ ,  $\frac{|\Sigma|}{a^3} \rightarrow 0$ . To apply dominated convergence, we use the following bounds which follow immediately from the above expression

$$\left| \frac{|\Sigma|}{a^3} \right| < \frac{1}{|z|^2 - 1}, \quad 1 < |z| < 2.$$

$$\left| \frac{|\Sigma|}{a^3} \right| < \frac{2}{|z|^{2n+2}} \leq \frac{2}{|z|^6}, \quad |z| > 2, n \geq 2.$$

Letting  $F_n(z)$  denote the integrand in (9), we have:

$$\begin{aligned} F_n(z) &= \frac{\exp\{-1/a\}}{a} \left[ \sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] \\ &\leq \exp\{-1/a\} \left[ \sqrt{\frac{|\Sigma|}{a^3}} + \sqrt{\pi} \frac{|\Re b|}{a^2} \right]. \end{aligned}$$

For  $|z| < 1$  we have:

$$F_n(z) \leq \exp\{-(1 - |z|^2)\} \left[ \frac{1}{\sqrt{1 - |z|^2}} + \sqrt{\pi} x \right],$$

which is integrable. If  $|z| > 1$  and  $n$  is large enough, we split the integral into regions  $1 < |z| < 2$  and  $|z| > 2$  and use the appropriate bounds from before. This justifies the use of the dominated convergence theorem.



In order to see the pointwise limit (10) of  $F_n(z)$ , we notice that for  $|z| < 1$ , we have (as  $n \rightarrow \infty$ ):

$$\begin{aligned}\sqrt{\frac{|\Sigma|}{a^3}} &\rightarrow \frac{1}{\sqrt{1-|z|^2}}, \\ a &\rightarrow \frac{1}{1-|z|^2}, \\ \frac{\Re\{b\}}{a^2} &\rightarrow x,\end{aligned}$$

and

$$\frac{\Re\{b\}}{\sqrt{a|\Sigma|}} \rightarrow x\sqrt{1-|z|^2}.$$

As pointed earlier, for  $|z| > 1$ , we have:

$$F_n(z) \rightarrow 0.$$

Combining these pointwise limits, we arrive at (10), and applying the dominated convergence theorem proves the formula (11) for the asymptotic expected length of a lemniscate generated by the Kac model.

#### 4. KOSTLAN POLYNOMIALS

In the next few sections we compare the average length of the lemniscate for different ensembles of random polynomials, starting with the Kostlan ensemble.

Consider a sequence of random polynomials whose coefficients are Kostlan random variables. Namely

$$P_n(z) = \sum_{k=0}^n a_{kn} z^k,$$

where  $a_{kn}$  are independent  $N_{\mathbb{C}}(0, \binom{n}{k})$ . Applying Theorem 4

$$(12) \quad \mathbb{E}|\Lambda| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[ \sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

where now for the Kostlan ensemble,

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

with

$$\begin{aligned}a &= K(z, z) = (1 + |z|^2)^n, \\ b &= n\bar{z}(1 + |z|^2)^{n-1}, \\ c &= n(n|z|^2 + 1)(1 + |z|^2)^{n-2}.\end{aligned}$$

This implies that

$$\begin{aligned} |\Sigma| &= ac - |b|^2 = n(1 + |z|^2)^{2n-2}, \\ \frac{|\Sigma|}{a^3} &= \frac{n}{(1 + |z|^2)^{n+2}}, \\ \frac{\Re\{b\}}{a^2} &= \frac{nx}{(1 + |z|^2)^{n+1}}, \\ \frac{|\Re b|}{\sqrt{a|\Sigma|}} &= \frac{nx^2}{(1 + |z|^2)^n}. \end{aligned}$$

Substituting these expressions into (12), we obtain

$$\mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \exp\left(-\frac{1}{(1 + |z|^2)^n}\right) [I_{1n}(z) + I_{2n}(z)] dA(z)$$

where  $I_{1n}(z) = \sqrt{\frac{n}{(1 + |z|^2)^{n+2}}} \exp\left(-\frac{nx^2}{(1 + |z|^2)^n}\right)$  and  $I_{2n}(z) = \sqrt{\pi} \frac{nx}{1 + |z|^2} \operatorname{erf}\left\{\sqrt{n}x/(1 + |z|^2)^{n/2}\right\}$

Converting the above integral into polar coordinates  $(r, \theta)$ , followed by the substitution  $r = \sqrt{\frac{t}{n}}$  leads us to

$$\begin{aligned} \mathbb{E}|\Lambda_n| &= \sqrt{\pi} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{1}{(1 + t/n)^n}\right) [J_{1n}(t, \theta) + J_{2n}(t, \theta)] dt d\theta, \\ J_{1n}(t, \theta) &= \sqrt{\frac{1}{n(1 + t/n)^{n+2}}} \exp\left(-\frac{t \cos^2(\theta)}{(1 + t/n)^n}\right) \\ J_{2n}(t, \theta) &= \sqrt{\pi} \frac{\sqrt{t} \cos(\theta)}{\sqrt{n}(1 + t/n)} \operatorname{erf}\left\{\sqrt{t} \cos(\theta)/(1 + t/n)^{n/2}\right\}. \end{aligned}$$

Removing a factor of  $1/\sqrt{n}$  from the  $J_{in}$ , we see that the resulting integral has a limit as  $n \rightarrow \infty$ . Namely, we have the following result

$$\sqrt{n} \mathbb{E}|\Lambda_n| \rightarrow I \quad \text{as } n \rightarrow \infty,$$

where  $I$  is the constant given by

$$I = \sqrt{\pi} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{1}{e^t}\right) \left[ \sqrt{\frac{1}{e^t}} \exp\left(-\frac{t \cos^2(\theta)}{e^t}\right) + \sqrt{\pi} \sqrt{t} \cos(\theta) \operatorname{erf}\left\{\sqrt{t} \cos(\theta)/e^{t/2}\right\} \right] dt d\theta.$$

## 5. WEYL POLYNOMIALS

In this section we consider Weyl polynomials defined by  $P_n(z) = \sum_{k=0}^n a_k z^k$  where  $a_k$  are independent random variables with  $a_k \sim N_{\mathbb{C}}(0, \frac{1}{k!})$ .

One can check easily that now the covariance matrix has entries given by

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

with

$$\begin{aligned} a &= \sum_{k=0}^n |z|^{2k}/k!, \\ b &= \bar{z} \sum_{k=1}^n |z|^{2k-2}/(k-1)!, \\ c &= \sum_{k=1}^n \frac{k^2}{k!} |z|^{2k-2}. \end{aligned}$$

Applying Theorem 4, we obtain

$$(13) \quad \mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[ \sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

All the quantities above have finite limits as  $n \rightarrow \infty$ . For instance  $a \rightarrow \exp(|z|^2)$ ,  $b \rightarrow \bar{z} \exp(|z|^2)$ , and  $c \rightarrow (1 + |z|^2) \exp(|z|^2)$ . Also, dominated convergence is easy to verify here. Taking the limit as  $n \rightarrow \infty$  in (13), we obtain

$$\mathbb{E}|\Lambda_n| \rightarrow L,$$

where

$$L = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{e^{|z|^2}}\}}{e^{|z|^2}} \left[ \sqrt{e^{|z|^2}} \exp\left\{-\frac{x^2}{e^{|z|^2}}\right\} + \sqrt{\pi} x \operatorname{erf}\left\{x/e^{|z|^2/2}\right\} \right] dA(z).$$

## 6. RECIPROCAL BINOMIAL DISTRIBUTION

Consider the polynomial ensemble  $P_n(z) = \sum_{k=0}^n a_{nk} z^k$ , where  $a_{nk}$  are independent random variables with  $a_{nk} \sim N_{\mathbb{C}}(0, \frac{1}{\binom{n}{k}})$ .

In this case, the entries  $a, b$  and  $c$  the entries of the covariance matrix are given as follows.

$$\begin{aligned} a &= \sum_{k=0}^n \frac{|z|^{2k}}{\binom{n}{k}}, \\ b &= \bar{z} \sum_{k=1}^n \frac{k|z|^{2k-2}}{\binom{n}{k}}, \\ c &= \sum_{k=1}^n \frac{k^2}{\binom{n}{k}} |z|^{2k-2}. \end{aligned}$$

.

Theorem 4 gives,

$$\mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[ \sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

We consider now asymptotically (with  $n$ ) the contribution of this integral from  $|z| < 1$  and  $|z| > 1$ . If  $|z| < 1$ , then we observe from the expressions for  $a, b$  and  $c$  that

$$\begin{aligned} a &= 1 + \frac{|z|^2}{n} + o(1), \\ b &= \frac{\bar{z}}{n} \left( 1 + \frac{4}{n-1} |z|^2 + o(1) \right), \\ c &= \frac{1}{n} \left( 1 + \frac{8}{n-1} |z|^2 + o(1) \right). \end{aligned}$$

This yields  $|\Sigma| = ac - |b|^2 = \frac{1}{n} (1 + o(1))$ ,  $\frac{|\Re b|}{a} = \frac{x}{n} (1 + o(1))$  and finally  $\frac{|\Re b|^2}{a|\Sigma|} = \frac{x^2}{n} (1 + o(1))$ . This implies that the integral for  $|z| < 1$  is of order  $\sqrt{\frac{1}{n}} (1 + o(1))$ . So  $\sqrt{n}\mathbb{E}|\Lambda_n|$  has a finite limit for  $z$  in the unit disc.

We next claim that asymptotically, the integral over  $|z| > 1$  goes to 0 (after a scaling by  $\sqrt{n}$ ). Indeed, notice then that

$$\begin{aligned} a &= |z|^{2n} \left( 1 + \frac{1}{n|z|^2} + o(1) \right), \\ b &= n\bar{z}|z|^{2n-2} \left( 1 + \frac{n-1}{n^2|z|^2} + o(1) \right), \\ c &= n^2|z|^{2n-2} \left( 1 + \frac{(n-1)^2}{n^3|z|^2} + o(1) \right). \end{aligned}$$

From here we can deduce that  $|\Sigma| = \frac{|z|^{4n-4}}{n} (1 + o(1))$ . This gives that

$$\sqrt{\frac{|\Sigma|}{a^3}} = \sqrt{\frac{1}{n}} \frac{1}{|z|^{n+2}} (1 + o(1)),$$

$$\frac{|\Re b|}{a^2} = \frac{n|x|}{|z|^{2n+2}} (1 + o(1)).$$

The pointwise limit of the integrand (even if we scale it by  $\sqrt{n}$ ) is clearly 0 and because of power decay, dominated convergence holds. So the contribution from the exterior of the unit disc to the integral is negligible. Ultimately we get

$$\sqrt{n}\mathbb{E}|\Lambda_n| \rightarrow C,$$

where the constant  $C$  is given by an integral over  $|z| < 1$  independent of  $n$ .

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